ON FINITE SIMPLE GROUPS OF ESSENTIAL DIMENSION 3

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ABSTRACT. We show that the only finite simple groups of essential dimension 3 (over $\mathbb{C}$) are $\mathfrak{A}_6$ and possibly $\mathrm{PSL}_2(\mathbb{F}_{11})$. This is an easy consequence of the classification by Prokhorov of rationally connected threefolds with an action of a simple group.

INTRODUCTION

Let G be a finite group, and X a complex projective variety with a faithful action of G. We will say that X is a *linearizable* if there exists a complex representation V of G and a rational dominant G-equivariant map $V \dashrightarrow X$ (such a map is called a *compression* of V). The *essential dimension* $\mathrm{ed}(G)$ of G (over $\mathbb{C}$) is the minimal dimension of all linearizable G-varieties. We have to refer to [BR] for the motivation behind this definition; in a very informal way, $\mathrm{ed}(G)$ is the minimum number of parameters needed to define all Galois extensions L/K with Galois group G and $K \supset \mathbb{C}$.

The groups of essential dimension 1 are the cyclic groups and the diedral group D_n, n odd [BR]. The groups of essential dimension 2 are classified in [D2]; the list is already large, and such classification becomes probably intractable in higher dimension. However the *simple* (finite) groups in the list are only $\mathfrak{A}_5$ and $\mathrm{PSL}_2(\mathbb{F}_7)$. In this note we try to go one step further:

Proposition. *The simple groups of essential dimension* 3 *are* $\mathfrak{A}_6$ *and possibly* $\mathrm{PSL}_2(\mathbb{F}_{11})$.

The result is an easy consequence of the remarkable paper of Prokhorov [P], who classifies all rationally connected threefolds admitting the action of a simple group. We can rule out most of the groups appearing in [P] thanks to a simple criterion [RY]: if a G-variety X is linearizable, any abelian subgroup of G must fix a point of X. Unfortunately this criterion does not apply to $\mathrm{PSL}_2(\mathbb{F}_{11})$, whose only abelian subgroups are cyclic or isomorphic to $(\mathbb{Z}/2)^2$.

1. PROKHOROV'S LIST

Let G be a finite simple group with $\mathrm{ed}(G) = 3$. By definition there exists a linearizable projective G-threefold X. This implies in particular that X is rationally connected. Such pairs (G, X) have been classified in [P]: up to conjugation, we have the following possibilities:

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(1) $G = \mathrm{SL}_2(\mathbb{F}_8)$ acting on a Fano threefold $X \subset \mathbb{P}^8$;
(2) $G = \mathfrak{A}_5, \mathfrak{A}_6, \mathfrak{A}_7, \mathrm{PSL}_2(\mathbb{F}_7), \mathrm{PSL}_2(\mathbb{F}_{11})$, or $\mathrm{PSp}_4(\mathbb{F}_3)$.

The groups $\mathfrak{A}_5, \mathfrak{A}_6, \mathfrak{A}_7$ have essential dimension 2,3 and 4 respectively, and $\mathrm{PSL}_2(\mathbb{F}_7)$ has essential dimension 2 [D1]. We are not able to settle the case $G = \mathrm{PSL}_2(\mathbb{F}_{11})$ (see §3). As for $\mathrm{PSp}_4(\mathbb{F}_3)$, we have:

Proposition 1. *The essential dimension of* $\mathrm{PSp}(4, \mathbb{F}_3)$ *is* 4.

Proof : The group $\mathrm{Sp}(4, \mathbb{F}_3)$ has a linear representation on the space W of functions on $\mathbb{F}_3^2$, the *Weil representation*, for which we refer to [AR], Appendix I. This representation splits as $W = W^+ \oplus W^-$, the spaces of even and odd functions; we have $\dim W^+ = 5$, $\dim W^- = 4$. The central element $(-I)$ of $\mathrm{Sp}(4, \mathbb{F}_3)$ acts on W by ${}^{(-I)}F(x) = F(-x)$, hence it acts trivially on W^+, and as $-\mathrm{Id}$ on W^-. Thus we get a faithful representation of $\mathrm{PSp}(4, \mathbb{F}_3)$ on W^+, with a compression to $\mathbb{P}(W^+) \cong \mathbb{P}^4$, hence $\mathrm{ed}(\mathrm{PSp}(4, \mathbb{F}_3)) \leq 4$.

To prove that we have equality, we observe[1] that $\mathrm{PSp}(4, \mathbb{F}_3)$ contains a subgroup isomorphic to $(\mathbb{Z}/2)^4$. One way to see this is to use the isomorphism $\mathrm{PSp}(4, \mathbb{F}_3) \cong \mathrm{SO}^+(5, \mathbb{F}_3)$: the group of diagonal matrices with entries ± 1 and determinant 1 is contained in $\mathrm{SO}^+(5, \mathbb{F}_3)$, and isomorphic to $(\mathbb{Z}/2)^4$. By [BR] we have

$$\mathrm{ed}(\mathrm{PSp}(4, \mathbb{F}_3)) \geq \mathrm{ed}((\mathbb{Z}/2)^4) = 4 . \qquad \blacksquare$$

2. The group $\mathrm{SL}_2(\mathbb{F}_8)$

It remains to prove that the pair $(\mathrm{SL}_2(\mathbb{F}_8), X)$ mentioned in (1) is not linearizable. To do this we will use the following criterion ([RY], Appendix):

Lemma 1. *If (G, X) is linearizable, every abelian subgroup of G has a fixed point in X.*

Proposition 2. *The essential dimension of* $\mathrm{SL}_2(\mathbb{F}_8)$ *is* ≥ 4.

The group $\mathrm{SL}_2(\mathbb{F}_8)$ has a representation of dimension 7, hence its essential dimension is ≤ 6 – we do not know its precise value.

Proof : The group $\mathrm{SL}_2(\mathbb{F}_8)$ acts on a rational Fano threefold $X \subset \mathbb{P}^8$ in the following way [P]. Let U be an irreducible 9-dimensional representation of $\mathrm{SL}_2(\mathbb{F}_8)$; there exists a non-degenerate invariant quadratic form q on U, unique up to a scalar. Then $SL_2(\mathbb{F}_8)$ acts on the orthogonal Grassmannian $\mathbb{G}_{\mathrm{iso}}(4, U)$ of 4-dimensional isotropic subspaces of U. This Grassmannian admits a $O(q)$-equivariant embedding into $\mathbb{P}^{15}$, given by the half-spinor representation [M]. The threefold X is the intersection of $\mathbb{G}_{\mathrm{iso}}(4, U)$ with a subspace $\mathbb{P}^8 \subset \mathbb{P}^{15}$ invariant under $\mathrm{SL}_2(\mathbb{F}_8)$.

Let $N \subset \mathrm{SL}_2(\mathbb{F}_8)$ be the subgroup of matrices $\begin{pmatrix} I & a \\ 0 & I \end{pmatrix}$, $a \in \mathbb{F}_8$. We will show that N has no fixed point in $\mathbb{G}_{\mathrm{iso}}(4, U)$, and therefore in X.

[1] I am indebted to A. Duncan for this observation.

Let χ_U be the character of the representation U. We have $\chi_U(n) = 1$ for $n \in N$, $n \neq 1$ (see for instance [C], 2.7). It follows that U restricted to N is the sum of the regular representation and the trivial one; in other words, as a N-module we have

$$U = \mathbb{C}_1^2 \oplus \sum_{\substack{\lambda \in \hat{N} \\ \lambda \neq 1}} \mathbb{C}_\lambda \ ,$$

where $\mathbb{C}_\lambda$ is the one-dimensional representation associated to the character λ. The subspaces $\mathbb{C}_\alpha$ and $\mathbb{C}_\beta$ must be orthogonal for $\alpha \neq \beta$; since q is non-degenerate, its restriction to each $\mathbb{C}_\lambda$ ($\lambda \neq 1$) and to $\mathbb{C}_1^2$ must be non-degenerate.

Now any vector subspace $L \subset U$ fixed by N must be the sum of some of the $\mathbb{C}_\lambda$, for $\lambda \neq 1$, and of some subspace of $\mathbb{C}_1^2$; this implies that L cannot be isotropic as soon as $\dim L \geq 2$. Hence N has no fixed point on $\mathbb{G}_{\mathrm{iso}}(4, U)$, and X is not linearizable by Lemma 1. ∎

3. About $\mathrm{PSL}_2(\mathbb{F}_{11})$

The Weil representation W^- of $\mathrm{SL}_2(\mathbb{F}_{11})$ factors through $\mathrm{PSL}_2(\mathbb{F}_{11})$, hence provides a 5-dimensional representation of the latter group; thus its essential dimension is 3 or 4. According to [P] there are two rationally connected threefolds with an action of $\mathrm{PSL}_2(\mathbb{F}_{11})$, the Klein cubic $X^{\mathrm{k}} \subset \mathbb{P}^4$ given by $\sum_{i \in \mathbb{Z}/5} X_i^2 X_{i+1} = 0$ and a Fano threefold $X^{\mathrm{a}} \subset \mathbb{P}^9$ of degree 14, birational to X^{k}. The group $\mathrm{PSL}_2(\mathbb{F}_{11})$ has order $660 = 2^2.3.5.11$; its abelian subgroups are cyclic, except the 2-Sylow subgroups which are isomorphic to $(\mathbb{Z}/2)^2$. A finite order automorphism of a rationally connected variety has always a fixed point (for instance by the holomorphic Lefschetz formula); one checks easily that a 2-Sylow subgroup of $\mathrm{PSL}_2(\mathbb{F}_{11})$ has a fixed point on both X^{k} and X^{a}. So lemma 1 does not apply, and another approach is needed.

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